OPEN PROBLEMS AND CONJECTURES IN DIFFERENCE EQUATIONS

By

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This talk is dedicated to Professors Vasilis Staikos and Yiannis Sficas for their profound contributions to the University of Ioannina.

1. What do the following equations have in common?

$$x_{n+1} = \frac{1}{x_n}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{1}{x_n x_{n-1}}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{1}{x_{n-1}}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{1 + x_n + x_{n-1}}{x_{n-2}}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_{n-1}}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_n x_{n-1}}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{\max\{x_n, 1\}}{x_n^2 x_{n-1}}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{\max\{x_n^2, 1\}}{x_n x_{n-1}}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{\max\{x_n^2, 1\}}{x_n^3 x_{n-1}}, \quad n = 0, 1, \ldots$$

The answer is that every solution of each of the above equations is periodic with the same period. What is the period? What is it that makes every solution of an equation periodic with the same period?

Is there a necessary and/or sufficient condition that can be used to determine this in an easily verifiable way?

2. Is There a Periodic Pattern Here?

$$x_{n+1} = \frac{1}{x_n}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \frac{1 + x_n + x_{n-1}}{x_{n-2}}, \quad n = 0, 1, \ldots$$

Unfortunately, there is no pattern. Why?

3. Can We Trust the Computer?

The **gingerbreadman difference equation** is the piecewise linear difference equation

$$x_{n+1} = |x_n| - x_{n-1} + 1, \quad n = 0, 1, \ldots \tag{1}$$

which was investigated by Devaney (see [2]) and was shown to be chaotic in certain regions and stable in others. The name of this equation is due to the fact that the orbits of certain points in the plane fill a region that looks like a "gingerbreadman".

If you use a computer to plot the orbit of the solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(1) with initial conditions

$$(x_{-1}, x_0) = \left(-\frac{1}{10}, 0\right)$$

the computer may predict that after 100,000 iterations, **the solution is still not periodic.** Although a computer may be fooled due to round-off and truncation errors, one can show that the orbit of the solution of Eq.(1) with initial condition

$$(x_{-1}, x_0) = \left(-\frac{1}{10}, 0\right)$$

is **periodic with period 126.** So, we cannot trust the computer. However, the computer is an indispensable tool in our investigations.

It is interesting to note that the gingerbreadman difference equation is a special case of the **max difference equation**

$$x_{n+1} = \frac{\max\{x_n^2, A\}}{x_n x_{n-1}}, \quad n = 0, 1, \ldots \tag{2}$$

Indeed, the change of variables

$$x_n = \begin{cases} A^{\frac{1+y_n}{2}} & if \quad A > 1 \\ e^{\frac{y_n}{2}} & if \quad A = 1 \\ A^{\frac{-1+y_n}{2}} & if \quad 0 < A < 1 \end{cases}$$

reduces Eq.(2) to the difference equation

$$y_{n+1} = |y_n| - y_{n-1} + \delta, \quad n = 0, 1, \ldots$$

where

$$\delta = \begin{cases} -1 & if \quad A > 1 \\ 0 & if \quad A = 1 \\ 1 & if \quad A < 1 \end{cases}$$

Note that Eq.(2) with

$$A \in (0, 1)$$

reduces to the gingerbreadman difference equation (1).

When

$$A = 1$$

Eq.(2) reduces to the equation

$$y_{n+1} = |y_n| - y_{n-1}, \quad n = 0, 1, \ldots \quad . \tag{3}$$

Note that every solution of Eq.(4) is periodic with period 9.

Open Problem 1. What is the set of initial conditions

$$(x_{-1}, x_0) \in (0, \infty) \times (0, \infty)$$

through which the solutions of Eq.(1) are periodic?

Are there values of A, other than $A = 1$, for which every solution of Eq.(2) is periodic with the same period?

What do the solutions of Eq.(2) do for values of A not equal to 1?

Conjecture 1. Assume $k, A \in [0, \infty)$.

Show that every positive solution of the equation

$$x_{n+1} = \frac{\max\{x_n^k, A\}}{x_{n-1}}, \quad n = 0, 1, \ldots$$

is bounded if and only if $k \in [0, 2)$ and every positive solution of the equation

$$x_{n+1} = \frac{\max\{x_n^k, A\}}{x_n x_{n-1}}, \quad n = 0, 1, \ldots$$

is bounded if and only if $k \in [0, 3)$.

4. The Simplest and Most Difficult Open Problem and Conjecture for Y2K+...

The simplest and most difficult open problem and conjecture for several years now are the following:

Conjecture 2. Assume $\alpha, \beta \in (0, \infty)$. Show that every positive solution of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n}{1 + x_{n-1}}, \quad n = 1, 2, \ldots$$

has a finite limit. (See [7] and [8]).

Open Problem 2. Exhibit all initial points $(x_{-1}, x_0) \in R^2$ through which the equation

$$x_{n+1} = -1 + \frac{x_{n-1}}{x_n}$$

is well defined for all $n \geq 0$ and investigate the character of its solutions. (See [1]).

5. What have we learned from the following two equations?

$$x_{n+1} = \frac{x_n + x_{n-1}}{2}, \quad n = 0, 1, \ldots \tag{4}$$

$$x_{n+1} = \sqrt{x_n} + \sqrt{x_{n-1}}, \quad n = 0, 1, \ldots \tag{5}$$

These equations have taught us some deep global attractivity results for difference equations. (See [3]-[9]).

Both of these equations are of the form

$$x_{n+1} = f(x_n, x_{n-1}, \quad n = 0, 1, \ldots \tag{6}$$

where the function f is increasing in both variables. Furthermore, for Eq.(6) the function f has the property that

$$f(x, x) = x \quad \text{for every } x$$

while for Eq.(5) the function f satisfies the *negative feedback property*

$$(f(x, x) - x)(x - \overline{x}) < 0 \quad \text{for } x \neq \overline{x}$$

where $\overline{x}$ is the equilibrium of the equation.

Under either of the assumptions above, every solution of Eq.(6) has a limit.

These two global results have been of paramount importance in establishing the global character of solutions of some rational difference equations and in several applications.

What do the following models have in common with the difference equation

$$x_{n+1} = \sqrt{x_n} + \sqrt{x_{n-1}}, \quad n = 0, 1, \ldots?$$

1. The Perennial Grass Models:

$$(i) \quad x_{n+1} = ax_n + be^{-x_n}, \quad n = 0, 1, \ldots$$
$$(ii) \quad x_{n+1} = ax_n + (b + cx_{n-1})e^{-x_n}, \quad n = 0, 1, \ldots$$

where $a, c \in (0, 1)$ and $b \in (0, \infty)$.

2. The Age-Structured Population Model:

$$x_{n+1} = (ax_n + bx_{n-1})e^{-x_n}, \quad n = 0, 1, \ldots$$

where $a \in (0, 1)$ and $b \in (0, \infty)$.

3. The Mosquito Model

$$x_{n+1} = (ax_n + bx_{n-1}e^{-x_{n-1}})e^{-x_n}, \quad n = 0, 1, \ldots$$

where $a \in (0, 1)$ and $b \in (0, \infty)$.

4. The Beddington-Holt Stock-Recruitment Model

$$x_{n+1} = ax_n + \frac{bx_{n-1}}{1 + cx_{n-1} + dx_n}, \quad n = 0, 1, \ldots$$

where $a \in (0, 1)$ and $b, c, d \in (0, \infty)$.

5. The Larval-Pupal-Adult (LPA) Flour Beetle Model

$$x_{n+1} = ax_n + bx_{n-2}e^{-c_1x_{n-2} - c_2x_n}, \quad n = 0, 1, \ldots$$

where $a \in (0, 1)$, $b \in (0, \infty)$, $c_1, c_2 \in [0, \infty)$ with $c_1 + c_2 > 0$.

6. Convergence to Periodic Solutions

The following are difference equations with the property that every positive solution converges to a periodic solution, as indicated. At present we are unable to determine the limit in terms of the initial conditions of the solution. This is a problem of paramount importance and great difficulty.

$$x_{n+1} = 1 + \frac{x_{n-1}}{x_n} \to P_2$$

$$x_{n+1} = 1 + \frac{x_{n-1}}{x_{n-2}} \to P_2$$

$$x_{n+1} = \frac{A}{x_n} + \frac{B}{x_{n-2}} \to P_2 \quad ; A, B \in (0, \infty)$$

$$x_{n+1} = \frac{x_n + x_{n-2}}{x_{n-1}} \to P_4$$

$$x_{n+1} = \frac{1 + x_{n-2}}{x_n} \to P_5$$

$$x_{n+1} = \frac{1 + x_n}{x_{n-1} + x_{n-2}} \to P_6$$

$$x_{n+1} = \frac{1 + x_n + x_{n-k}}{x_{n-(k-1)}} \to P_{2k}$$

$$x_{n+1} = 1 + \frac{x_{n-k}}{1 + x_n + \cdots + x_{n-(k-1)}} \to P_{(k+1)}$$

$$x_{n+1} = \frac{1 + x_{n-1}}{1 + x_n + x_{n-2}} \to P_2$$

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} + x_{n-2}} \to P_2$$

$$x_{n+1} = x_{n-1} e^{-x_n} \to P_2$$

$$\left.\begin{array}{rcl} A_{n+1} & = & J_n \\ J_{n+1} & = & A_n e^{r-(A_n+J_n)} \end{array}\right\} \to P_2; \quad r \geq 0$$

$$\left.\begin{array}{rcl} x_{n+1} & = & \frac{a_{11}}{x_n} + \frac{a_{12}}{y_n} \\ y_{n+1} & = & \frac{a_{21}}{x_n} + \frac{a_{22}}{y_n} \end{array}\right\} \to P_2; \quad a_{ij} \in (0, \infty)$$

$$x_{n+1} = \frac{1}{x_n x_{n-1}} + \frac{1}{x_{n-3} x_{n-4}} \to P_3$$

7. What do the Following Equations Have in Common?

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{A}{x_{n-1}}\right\}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \max\left\{\frac{A_0}{x_n}, \frac{A_1}{x_{n-1}}, \cdots, \frac{A_k}{x_{n-k}}\right\}, \quad n = 0, 1, \ldots$$

$$x_{n+1} = \max\left\{\frac{1}{x_n}, \frac{1}{x_{n-1}}, \cdots, \frac{1}{x_{n-k}}\right\}, \quad n = 0, 1, \ldots$$

The answer is that every solution of each of the above equations is eventually periodic. What is the period in each case? See [4].

8. In the Spirit of the $(3x+1)$ Conjecture

The $(3x+1)$ conjecture states that every solution of the difference equation

$$x_{n+1} = \begin{cases} \frac{3x_n+1}{2} & if \quad x_n \text{ is odd} \\ \frac{x_n}{2} & if \quad x_n \text{ is even} \end{cases} \qquad n = 0, 1, \ldots$$

with initial condition

$$x_0 \in \{1, 2, \ldots\}$$

is eventually the 2-cycle $(1, 2)$.

On the other hand, if

$$x_0 \in \{\ldots, -2, -1\}$$

then it is conjectured that every solution is eventually either:

The one-cycle

$$(-1),$$

the three-cycle

$$(-5, -7, -10)$$

or the eleven-cycle

$$(-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34)$$

Conjecture 3. Assume that the initial conditions are integers with greatest common divisor equal to 1.

Then show that every solution of the difference equation

$$x_{n+1} = \begin{cases} \frac{x_n - x_{n-1}}{2} & if \quad (x_n + x_{n-1}) \text{ is even} \\ x_n + x_{n-1} & if \quad (x_n + x_{n-1}) \text{ is odd} \end{cases} \qquad n = 0, 1, \ldots$$

is eventually:

$(0, 1, 1)$, $(0, -1, -1)$, or $(3, 2, 5, 7, 1, -3, -2, -5, -7, -1)$

(See [4]).

Similar problems are of interest for the equation

$$x_{n+1} = \begin{cases} \frac{x_n - x_{n-1}}{3} & if \quad 3 \text{ divides } x_n + x_{n-1} \\ x_n + x_{n-1}, & \text{otherwise} \end{cases},$$

for $n = 0, 1, \ldots$, where x_{-1} and x_0 are integers.

Conjecture 4. The following statements are true:

(a) Every positive solution which is not eventually a three-cycle converges to ∞.
(b) There exist unbounded solutions.

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